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TECHNICAL REPORT 2.1

"ROTATION DESIGNS FOR SAMPLING

ON REPEATED OCCASIONS"

by

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ROTATION DESIGNS FOR SAMPLING ON REPEATED OCCASIONS*

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1. INTRODUCTION

There are many studies, notably in sociological and economic research, which are concerned with estimating characteristics of a population on 'repeated occasions' in order to measure time-trends as well as the current values of characteristics as a time series. For example, the Current Population Survey (C.P.S.) of the Bureau of the Census is, among other things, concerned with estimating monthly and yearly changes in the level of unemployment. Estimation problems in time series analysis have, of course, received considerable attention for some time but most of these studies are devoted to the estimation of parameters in time series models usually involving the concepts of infinite populations. On the other hand, most of the distribution free theory of sampling <u>finite</u> populations is concerned with estimates applying to a population at <u>one</u> particular <u>time</u>, and applies to so-called 'one shot surveys'. In this study we attempt a combination of finite population sampling with time series analysis. This activity, sometimes referred to as 'sampling on repeated occasions', has already received some attention although some of the references we discuss below are only concerned with infinite population models.

When the same population is sampled repeatedly, the opportunities for a flexible sample design are increased. For example, on the h^{th} occasion we may have parts of the sample that are matched with the $(h-1)^{th}$ occasion, parts that are matched with both the $(h-1)^{th}$ and the $(h-2)^{th}$ occasions,

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and soon. Such a method of partial matching has been termed "sampling on successive occasions with partial replacement of units" ([5]), "rotation sampling" ([2]), and "sampling for a time series" ([3]). The optimum replacement policy for two occasions has been investigated by Jessen [4] and the general problem of replacement for more than two occasions has been examined by Yates [7], Patterson [5], Eckler [2], and several others. However, the theory has been almost exclusively confined to infinite populations. A good summary of these papers is given by Cochran [1].

In the Current Population Survey conducted monthly by the U. S. Bureau of the Census (Hansen et al. [3]), a rotation sampling design is imposed within each primary mainly for the purpose of reducing response resistance (which may occur if the same panel of households is interviewed indefinitely), and to reduce the within primary component of variance of the estimates under certain circumstances. The rotation pattern is as follows: eight systematic sub-samples (rotation groups) of segments are identified for each sample. A given rotation group stays in the sample for four consecutive months, leaves the sample during the eight succeeding months, and then returns for another four consecutive months. It is then dropped from the sample. Under this system of sampling, 75 per cent of the sample segments are common between consecutive months and 50 per cent are common between the same months of two consecutive years.

The composite estimator of the population mean, $\bar{X}_0$, of the current month (denoted by 0) in this survey is

$$\bar{x}'_0 = Q(\bar{x}'_{-1} + \bar{x}_{0,-1} - \bar{x}_{-1,0}) + (1 - Q)\,\bar{x}_0 \tag{1}$$

where Q is a constant weight factor with $0 \le Q < 1$, $\bar{x}_0$ is the estimator based on the entire sample for the current month, 0, $\bar{x}_{0,-1}$ is the estimator for the current month but based on the sample segments common to both months 0 and -1, $\bar{x}_{-1,0}$ is the estimator for the previous month, -1, but based on the sample segments common to both months 0 and -1, and $\bar{x}'_{-1}$ is the composite estimator for the previous month, -1. The composite estimator of the change, $\bar{X}_0 - \bar{X}_{-1}$, is

$$d'_0 = \bar{x}'_0 - \bar{x}'_{-1} . \tag{2}$$

The composite estimators take advantage of the information obtained on previous occasions as well as the information from the current occasion, and results in smaller variances for both the current estimate and the estimate of change for most of the characteristics; the larger gains are usually achieved for the estimate of change. Hansen <u>et al</u>. [3] have given the variance of $\bar{x}'_0$ and d'_0, for their particular rotation pattern, under some simplifying assumptions and investigated the reduction in the within primary component of the variance of $\bar{x}'_0$ and d'_0 for some alternative assumed correlations between months and for selected values of Q.

The purpose of the present study is to develop a unified finite population theory for the composite estimators $\bar{x}'_0$ and d'_0. We formulate a general rotation pattern and obtain the variance of the composite estimators explicitly. We also obtain the optimum values of Q under certain simplifying assumptions regarding the correlation pattern from occasion to occasion. The variance for the particular rotation pattern used by Hansen <u>et al</u>. can be obtained as a special case from the general results.

Onate [6], in developing multistage designs for the Philippine Statistical Survey of Households, divided each sample barrio (a second stage unit that corresponds to a township in the United States) into a small number of segments (less than 10) and specified a particular rotation pattern for the segments. This rotation pattern was mainly intended to reduce the response resistance of panel households, for it was observed that the households tended to become uncooperative during the third or fourth visit. Moreover, Onate developed a finite population theory for the composite estimator of the barrio total for his special rotation pattern (which is a special case of the general rotation pattern). We follow Onate's approach in developing a finite population theory for the general rotation pattern. We give here the results for single-stage designs, but the extension to multistage designs, etc., is relatively straightforward and therefore is not presented here.

2. THE GENERAL ROTATION PATTERN

We assume that the actual units in the population remain unchanged in time, and denote by N and n the population and sample sizes, respectively (which are the same for all occasions). Also, let N and n be multiples of $n_2 (\geq 1)$. Then the rotation pattern is as follows: A group of n_2 units stays in the sample for r occasions ($n = n_2 r$), leaves the sample for m occasions, comes back into the sample for another r occasion, then leaves the sample for m occasions, and so on. If a unit returns to the sample after having dropped out (k - 1) previous times from the sample, we say the unit is in the k^{th} cycle. We only consider the case $m \geq r$ here, for case $m < r$ is more complicated and less useful in practice. The maximum value of m is $r(\frac{N}{n} - 1)$ and, if m is less than the maximum value, it amounts to covering only a fraction, viz., $n_2 m + n$ of the N units in the rotating design.

<u>Illustration</u>. Let $N = 5$, $n = r = 2$ and $m = 3$. Figure 1 illustrates

the rotation pattern for a particular numbering of the 5 units in the population. There are 5! possible permutations corresponding to the 5! ways of numbering the units in the population. The numbers 1 to 5 are assigned to the 5 units in the population. These permutations provide the stochastic input into the sampling design since it amounts to saying that the sample consists of one permutation selected at random from the finite population of 5! permutations.

Figure 1. Rotation pattern with $N = 5$, $n = r = 2$ and $m = 3$.

Unit \ Visit	-6	-5	-4	-3	-2	-1	0
1	X				X	X	
2	X	X				X	X
3		X	X				X
4			X	X			
5				X	X		

In general, the finite population is composed of the N! possible permutations and the sample consists of one randomly selected permutation from this population.

3. THE COMPOSITE ESTIMATORS $\bar{x}_0'$ and d_0'

Let $x_{\alpha j}$ denote the value of the characteristics for the j^{th} unit on the α^{th} occasion, ($\alpha = 0, -1, \ldots, -u$; $j = 1, \ldots, N$), where $-u$ denotes the occasion at which sampling first takes place. We assume that u is large in order to simplify the derivation of the variance formulae.

The composite estimator of the current occasion population mean, $\bar{X}_0$, is

$$\bar{x}'_0 = Q(\bar{x}'_{-1} + \bar{x}_{0,-1} - \bar{x}_{-1,0}) + (1 - Q)\,\bar{x}_0 \tag{3}$$

where

$$\bar{x}_{\alpha,\alpha-1} = \sum_1^{n_1} x_{\alpha,j}/n_1,\ \bar{x}_{\alpha-1,\alpha} = \sum_1^{n_1} x_{\alpha-1,j}/n_1,\ \bar{x}_\alpha = \sum_1^{n} x_{\alpha,j}/n\ ,$$

$\bar{x}'_{-1}$ is the composite estimator for occasion -1, and $0 \leq Q < 1$.

Here $n_1 = n_2(r - 1)$ is the number of units matched between occasions $\alpha - 1$ and α. The composite estimator of the population change, $\bar{X}_0 - \bar{X}_{-1}$, is

$$d_0 = \bar{x}'_0 - \bar{x}'_{-1}\ . \tag{4}$$

Now $\bar{x}'_0$ can be written as

$$\bar{x}'_0 = \sum_{\alpha=0}^{-u} Q^{-\alpha} W_\alpha = \sum_{\alpha=0}^{-u} \sum_{k=1}^{N} w_{\alpha,k}\, x_{\alpha,k} \tag{5}$$

where

$$W_\alpha = Q(\bar{x}_{\alpha,\alpha-1} - \bar{x}_{\alpha-1,\alpha}) + (1 - Q)\,\bar{x}_\alpha \tag{6}$$

for $\alpha = 0, -1, \ldots, -(u - 1)$, $W_{-u} = \bar{x}_{-u}$ and the $w_{\alpha,k}$ are functions of Q, r and n_2. From (5) and (6) it may be verified that the weights $w_{\alpha,k}$ are as follows:

For the current occasion, $\alpha = 0$,

(a) $w_{0,k} = (1 - Q)/n$ for n_2 units (first visit of a cycle)

(b) $w_{0,k} = (1 - Q)/n + Q/n_1$ for n_2 units (second to r^{th} visit of a cycle) (7)

(c) $w_{0,k} = 0$ for $N - n$ units (not in the sample)

For occasions $\alpha = -1, \ldots, -(u - 1)$,

(a) $w_{\alpha,k} = \dfrac{Q^{-\alpha}(1 - Q)}{n} - \dfrac{Q^{-\alpha}}{n_1}$ for n_2 units (first visit of a cycle)

(b) $w_{\alpha,k} = \dfrac{Q^{-\alpha}(1 - Q)}{n} + \dfrac{Q^{-\alpha+1}}{n_1} - \dfrac{Q^{-\alpha}}{n_1}$ for $n_1 - n_2$ units (second to $(r-1)^{th}$ visit of a cycle) (8)

(c) $w_{\alpha,k} = \dfrac{Q^{-\alpha}(1 - Q)}{n} + \dfrac{Q^{-\alpha+1}}{n_1}$ for n_2 units (r^{th} visit of a cycle)

(d) $w_{\alpha,k} = 0$ for $N - n$ units (not in the sample)

For occasion -u, $w_{-u,k} = Q^u(1/n - 1/n_1)$ for n_1 units (first to $(r-1)^{th}$ visit), $w_{-u,k} = Q^u/n$ for n_2 units (r^{th} visit), and $w_{-u,k} = 0$ for N-n units (not in the sample). Therefore, the averages of the weights over the N! possible permutations are

$$E(w_{0,k}) = \frac{1}{N!} \sum_{1}^{N} (N-1)!\, w_{0,k} = \frac{1}{N} \sum_{1}^{N} w_{0,k} = \frac{1}{N}$$

and

$$E(w_{\alpha,k}) = \frac{1}{N!} \sum_{1}^{N} (N-1)!\, w_{\alpha,k} = \frac{1}{N} \sum_{1}^{N} w_{\alpha,k} = 0, \ \alpha < 0.$$

Hence, $\bar{x}_0'$ is an unbiased estimator of $\bar{X}_0$. Similarly, d_0' is an unbiased estimator of $\bar{X}_0 - \bar{X}_{-1}$.

4. VARIANCE OF $\bar{x}_0'$ AND OPTIMUM Q

4.1 Variance of $\bar{x}_0'$

Since u is assumed to be large, the variance obtained by replacing -u by $-\infty$ will be virtually the same as the true variance. Therefore, the variance of $\bar{x}_0'$, $V(\bar{x}_0')$, is

$$V(\bar{x}_0') = E(\bar{x}_0'^2) - \bar{X}_0^2$$

$$\doteq \sum_{\alpha=0}^{-\infty} \sum_{k=1}^{N} E(w_{\alpha,k}^2)\, x_{\alpha,k}^2 + \sum_{\substack{\alpha\neq\alpha' \\ =0}}^{-\infty} \sum_{k=1}^{N} E(w_{\alpha,k}\, w_{\alpha'k})\, x_{\alpha,k}\, x_{\alpha'k}$$

$$+ \sum_{\alpha=0}^{-\infty} \sum_{\substack{k\neq k' \\ =1}}^{N} E(w_{\alpha,k}\, w_{\alpha,k'})\, x_{\alpha,k}\, x_{\alpha,k'} + \sum_{\substack{\alpha\neq\alpha' \\ =0}}^{-\infty} \sum_{\substack{k\neq k' \\ =1}}^{N} E(w_{\alpha,k}\, w_{\alpha'k'}) \cdot x_{\alpha,k}\, x_{\alpha'k'} \tag{9}$$

Now, since $\sum_{1}^{N} w_{0,k} = 1$ and $\sum_{1}^{N} w_{\alpha,k} = 0$, $\alpha < 0$,

$$E(w_{0,k}\, w_{0,k'}) = \frac{1}{N(N-1)} - \frac{1}{N-1} E(w_{0,k}^2)$$

$$E(w_{\alpha,k}\, w_{\alpha,k'}) = -\frac{1}{N-1} E(w_{\alpha,k}^2), \ \alpha < 0 \tag{10}$$

and

$$E(w_{\alpha,k}\, w_{\alpha'k'}) = -\frac{1}{N-1} E(w_{\alpha,k}\, w_{\alpha'k});\ \alpha \neq \alpha' = 0, -1, \ldots, -\infty$$

Using (10) in (9), we have

$$V(\bar{x}_0') = S_0^2 \left[N\, E(w_{0,k}^2) - \frac{1}{N} \right] + N \sum_{\alpha=-1}^{-\infty} E(w_{\alpha,k}^2)\, S_\alpha^2$$

$$+ N \sum_{\alpha\neq\alpha'=0}^{-\infty} E(w_{\alpha,k}\, w_{\alpha' k})\, S_{\alpha,\alpha'} \qquad (11)$$

where S_α^2 is the mean square for the α^{th} occasion and $S_{\alpha,\alpha'}$ is the mean product between occasions α and α'. Note that the variance formula (11) is quite general and is applicable for __any__ set of weights $w_{\alpha,k}$ such that $\sum_1^N w_{\alpha,k} = 0$, $\alpha < 0$, and $\sum_1^N w_{0,k} = 1$.

For the general rotation pattern the weights $w_{0,k}$ are given by (7) and the weights $w_{\alpha,k}$ $(\alpha < 0)$ are given by (8). Using these weights, we evaluate $E(w_{\alpha,k}^2)$ and $E(w_{\alpha,k}\, w_{\alpha' k})$. Now

$$N\, E(w_{0,k}^2) = \sum_1^N w_{0,k}^2 = \frac{1}{n}\left(1 + \frac{n_2}{n_1} Q^2\right) \qquad (12)$$

and

$$N\, E(w_{\alpha,k}^2) = \sum_1^N w_{\alpha,k}^2 = \frac{n_2}{nn_1} Q^{-2\alpha}\left(Q^2 + 2\frac{n_2}{n_1} Q + 1\right),\ \alpha < 0.$$

Noting that $N\, E(w_{\alpha,k}\, w_{\alpha' k}) = \sum_1^N w_{\alpha,k}\, w_{\alpha' k}$, and using the general rotation pattern and the weights in (7) and (8), we obtain, after simplification, the expectations $E(w_{\alpha,k}\, w_{\alpha' k})$ given in Appendix A. For example, since n_1 units are common between occasions 0 and $-s(r+m)-1$, $s = 0, \ldots, \infty$,

$$N\, E\,(w_{0,k}\, w_{-s(r+m)-1,k}) = \sum_1^N w_{0,k}\, w_{-s(r+m)-1,k}$$

$$= Q^{s(r+m)+1} \left[n_2\left(\frac{1-Q}{n} - \frac{1}{n_1}\right)\left(\frac{1-Q}{n} + \frac{Q}{n_1}\right)\right.$$

$$\left. + (n_1 - n_2)\left(\frac{1-Q}{n} + \frac{Q}{n_1} - \frac{1}{n_1}\right)\left(\frac{1-Q}{n} + \frac{Q}{n_1}\right) \right]$$

$$= -\frac{n_2}{n^2}\left(1 + \frac{n_2}{n_1} Q\right)^2 Q^{s(r+m)+1} .$$

Substitution in (11) of the expectations (12) and those of Appendix A yields the variance formula in Appendix B. It is interesting to note from Appendix B that the effect of the finite population of size N on $V(\bar{x}_0')$ is simply to subtract $-S_0^2/N$ from the variance of $\bar{x}_0'$ derived by assuming N infinite.

The variance formula (B-1) is very unwieldy and, therefore, we consider two models for S_α^2 and $S_{\alpha,\alpha'}$ which simplify the variance formula and enable the evaluation of the optimum Q.

<u>Model 1</u>. We first consider an exponential correlation pattern. The model is

$$S_\alpha^2 = S_0^2,\ S_{\alpha,\alpha+\beta} = S_{0,\beta} \text{ and } S_{0,\beta} = \rho^{-\beta} S_0^2 \qquad (13)$$
$$\alpha,\ \beta = -1,\ -2,\ \ldots,\ -\infty.$$

This model has been used by Patterson [5] and others, and appears to be a satisfactory assumption in many practical situations. Equation (13) is a special case of a more general model $S_\alpha = S_0\ p^\alpha$ with $p = 1 + \delta$, where δ is a small positive number. Our calculations, however, have shown that the variance with $\delta = 0$ is practically the same as the variance with a small δ (say 0.03). Therefore, we confine our study here to the simpler model, (13). With this model, the variance of $\bar{x}_0'$, given in Appendix B, reduces to

$$V(\bar{x}_0') = \left(\frac{1}{n} - \frac{1}{N}\right) S_0^2 + \frac{2n_2^3\ Q\ S_0^2}{n^2 n_1^2 (1 - Q^2)(1 - Q\rho)^2 \quad 1 - (Q\rho)^{r+m}}$$
$$\Big\{ Q^2 [\ r\rho^2 - (r^2 + 1)\ \rho + r\ ] + Q\ [\ r(r - 1)\ \rho^2 - 2(r - 1)\ \rho + r(r - 1)]$$
$$- (r - 1)^2\ \rho + Q^r\ \rho^{r-1} \big(\ Q^2\ [\ - (r - 1)\ \rho^2 + r(r - 1)\ \rho]$$
$$+ Q\ [\ - (r^2 - 2r + 2)\ \rho^2 + 2r\ \rho - r^2] - (r - 1)\ \rho^2 + r(r - 1)\ \rho\ \big)$$
$$+ Q^m\ \rho^{m+1} \big(\ Q^3\ [- r^2\rho^2 + r(r + 1)\ \rho - r\ ] + Q^2\ [\ 2r(r - 1)\ \rho - (r-1)(r+1)]$$
$$+ Q\ [- r(r - 1)\ \rho - (r - 1)(r - 2)] + (r - 1)^2 \big) + Q^{m+r}\ \rho^{m+r} \big(\ Q^3\ [\ r(r - 1)\rho^2$$
$$- r(r - 1)\ \rho\ ] + Q^2 [\ r\ \rho^2 - (2r^2 - r + 1)\ \rho + r^2\ ] + Q\ [(r - 1)(r - 2)\ \rho$$
$$+ r(r - 1)] + (r - 1)\ \rho - r(r - 1) \big) \Big\}\ . \qquad (14)$$

For moderate and large values of m (which implies that N is large), the following formula for $V(\bar{x}_0')$, obtained from (14) by substituting $m = \infty$ and ignoring the finite population correction term (i.e., the term $-S_0^2/N$), is a good approximation:

$$V(\bar{x}_0') = \frac{S_0^2}{n} + \frac{2n_2^3 Q S_0^2}{n^2 n_1^2 (1 - Q^2)(1 - Q\rho)^2} \Big\{ Q^2 [r \rho^2 - (r^2 + 1)\rho + r]$$

$$+ Q[r(r - 1)\rho^2 - 2(r - 1)\rho + r(r - 1)] - (r - 1)^2 \rho + Q^r \rho^{r-1} .$$

$$(Q^2 [-(r - 1)\rho^2 + r(r - 1)\rho] + Q[-(r^2 - 2r + 2)\rho^2 + 2r\rho - r^2]$$

$$- (r - 1)\rho^2 + r(r - 1)\rho) \Big\} . \qquad (15)$$

Our calculations in sub-section 4.2 indicate that for moderate and large values of m, $V(\bar{x}_0')$ as obtained from (15) differs very little from that obtained from (14), and the optimum values of Q are unaffected by m in most cases.

The special case $x_{\alpha,k} = x_{\alpha' k}$, $\alpha, \alpha' = 0, -1, \ldots, -\infty$, provides a check on (14). Then $\bar{x}_0'$ reduces to

$$\bar{x}_0' = (1 - Q) \sum_{\alpha=0}^{-\infty} Q^{-\alpha} \bar{x}_\alpha$$

and

$$V(\bar{x}_0') = (1 - Q)^2 \Big[\sum_{\alpha=0}^{-\infty} Q^{-2\alpha} V(\bar{x}_\alpha) + 2 \sum_{\alpha \neq \alpha' = 0}^{-\infty} Q^{-\alpha} Q^{-\alpha'} \operatorname{Cov}(\bar{x}_{-\alpha}, \bar{x}_{-\alpha'}) \Big] . \qquad (16)$$

Now

$$V(\bar{x}_\alpha) = (\frac{1}{n} - \frac{1}{N}) S_0^2$$

$$\operatorname{Cov}(\bar{x}_\alpha, \bar{x}_{\alpha-s(r+m)-t}) = \Big[\frac{n_2(r - t)}{n^2} - \frac{1}{N} \Big] S_0^2 \quad \text{for } t < r$$

$$\operatorname{Cov}(\bar{x}_\alpha, \bar{x}_{\alpha-s(r+m)-t}) = - \frac{S_0^2}{N} \qquad \text{for } r \leq t \leq m$$

and

$$\operatorname{Cov}(\bar{x}_\alpha, \bar{x}_{\alpha-s(r+m)-m-t}) = (\frac{tn_2}{n^2} - \frac{1}{N}) S_0^2 \qquad \text{for } 1 \leq t \leq r.$$

Substituting these values into (16), we find, after simplification,

$$V(\bar{x}_0') = \left(\frac{1}{n} - \frac{1}{N}\right) S_0^2 - \frac{2n_2\, Q\, S_0^2}{n^2(1 - Q^2)(1 - Q^{r+m})} \left[1 - Q^r - Q^m + Q^{r+m} \right] . \quad (17)$$

The special case $x_{\alpha,k} = x_{\alpha' k}$ implies $\rho = 1$. With $\rho = 1$, (14) reduces to (17), thus providing a check.

The per cent gain in efficiency of $\bar{x}_0'$ over $\bar{x}_0$ is

$$\frac{V(\bar{x}_0) - V(\bar{x}_0')}{V(\bar{x}_0')} \times 100 \quad (18)$$

where

$$V(\bar{x}_0) = \left(\frac{1}{n} - \frac{1}{N}\right) S_0^2 . \quad (19)$$

Model 2. We now consider an arithmetic correlation pattern similar to that of Hansen et al. [3]. As mentioned earlier, the empirical results with Model 1, given in sub-section 4.2, indicate that $V(\bar{x}_0')$ computed from (14) with $m = \infty$ differs very little from $V(\bar{x}_0')$ with moderate m, and the optimum Q is unaffected in most cases. In order to simplify the discussion we, therefore, confine ourselves here to an arithmetic correlation pattern and to the variance of $\bar{x}_0'$ with $m = \infty$. The model is

$$S_\alpha^2 = S_0^2,\ S_{\alpha,\alpha+\beta} = S_{0,\beta},\ \alpha, \beta = -1, \ldots, -\infty;\ t = 1, 2, \ldots, r-1 \quad (20)$$

$$S_{0,\alpha} = [\,\rho + (\alpha + 1)\, d\,]\, S_0^2 \text{ for } -(\alpha + 1)\, d \leq \rho$$

$$= 0 \qquad \text{for } -(\alpha + 1)\, d > \rho$$

where d is a small positive number. Using (20) in the general variance formula, given in Appendix B, with $m = \infty$, and neglecting the finite population correction term, we obtain

$$V(\bar{x}_0') = \frac{S_0^2}{n} + \frac{2S_0^2}{nr(r-1)^2(1-Q^2)(1-Q)^3} \Big\{ Q^2 [\, r(r-1) - r(3r-4)Q + 3r(r-2)\, Q^2 - r(r-4)Q^3 - r\, Q^4] + Q\, \rho\, [\, -(r-1)^2 + (r-1)(2r-3)\, Q + (2r-3)\, Q^2 - (2r^2-4r-1)Q^3 + r(r-4)Q^4 + rQ^5 \,] + Q^2 d [\,(r-1)(r-2) - (r-2)^2 Q - (r^2-2r-2)Q^2 + r(r-4)Q^3 + rQ^4] \Big\}$$

$$+ \frac{2S_0^2 \, Q^{r+1} d}{nr(r-1)(1-Q^2)(1-Q)^3} \Big\{ - r(r-2) + (3r^2-8r+2)Q - (3r^2-10r+4)Q^2$$

$$+ (r^2-4r+2)Q^3 \Big\} + \frac{2S_0^2 \, Q^{r+1} \rho}{nr(1-Q^2)} \quad . \tag{21}$$

The per cent gain in efficiency of $\bar{x}_0'$ relative to $\bar{x}_0$ may be computed from (18) using (21) and (19) without the finite population correction.

4.2 Optimum Q and per cent gain in efficiency

Model 1. Using an IBM 7074 with a 20k memory we computed the per cent gain in efficiency from (14), (18) and (19) for $Q = 0.1(0.1)0.9$, $\rho = 0.5(0.1)0.9$, $r = 2(1)4(2)8$ and $m = sr$ with $s = 1(1)4, \infty$. In computing the per cent gain in efficiency we ignored the term $-S_0^2/N$ in (14) and (19), since it does not change the optimum value of Q. The effect of ignoring $-S_0^2/N$ is to decrease the per cent gain in efficiency. The optimum Q for each combination (r, ρ, m) is that value of Q which gives the maximum per cent gain in efficiency. The optimum Q given below is correct to only the first decimal place. However, it is quite satisfactory since small deviations from the true optimum lead only to a very small loss in efficiency. Table 1 gives the per cent gain in efficiency and the optimum Q for the selected values of r, m and ρ. Several interesting results emerge from Table 1. First, the value of $V(\bar{x}_0')$ for moderate m is virtually the same as that for $m = \infty$ obtained from the simplified formula (15). Secondly, the optimum Q is unaffected by m (except for the case $r = 4$ and $\rho = 0.9$ where the optimum Q deviates by 0.1 from $m = 4$ to $m = 8$). Thirdly, the optimum value of r is 2. However, if we are interested in estimating the change simultaneously then $r = 2$ may not be optimum. Moreover, other practical considerations may sometimes warrant the use of an r other than 2. Finally, the gains in efficiency of $\bar{x}_0'$ over $\bar{x}_0$ are only moderate, even with a fairly high ρ.

Table 1. Per cent gain in efficiency of $\bar{x}_0'$ over $\bar{x}_0$, and in parantheses optimum Q

m \ ρ	0.5	0.6	0.7	0.8	0.9
			r = 2		
2	5.2(0.2)	8.5(0.2)	14.2(0.3)	22.8(0.4)	38.8(0.5)
4	5.3(0.2)	8.7(0.2)	15.3(0.3)	26.7(0.4)	50.6(0.5)
8	5.3(0.2)	8.7(0.2)	15.3(0.3)	27.1(0.4)	55.7(0.6)
∞	5.3(0.2)	8.7(0.2)	15.3(0.3)	27.1(0.4)	56.2(0.6)
			r = 3		
3	4.2(0.2)	7.2(0.3)	12.0(0.4)	20.2(0.5)	36.6(0.6)
6	4.2(0.2)	7.3(0.3)	12.4(0.4)	22.3(0.5)	45.3(0.6)
9	4.2(0.2)	7.3(0.3)	12.4(0.4)	22.4(0.5)	46.7(0.6)
∞	4.2(0.2)	7.3(0.3)	12.4(0.4)	22.4(0.5)	47.0(0.6)
			r = 4		
4	3.3(0.2)	5.8(0.3)	9.8(0.4)	17.1(0.5)	32.4(0.6)
8	3.3(0.2)	5.8(0.3)	9.9(0.4)	17.8(0.5)	38.6(0.7)
12	3.3(0.2)	5.8(0.3)	9.9(0.4)	17.8(0.5)	39.8(0.7)
∞	3.3(0.2)	5.8(0.3)	9.9(0.4)	17.8(0.5)	40.0(0.7)
			r = 6		
6	2.3(0.3)	4.0(0.3)	6.7(0.4)	12.0(0.6)	26.0(0.7)
12	2.3(0.3)	4.0(0.3)	6.7(0.4)	12.2(0.6)	28.3(0.7)
18	2.3(0.3)	4.0(0.3)	6.7(0.4)	12.2(0.6)	28.5(0.7)
∞	2.3(0.3)	4.0(0.3)	6.7(0.4)	12.2(0.6)	28.5(0.7)
			r = 8		
8	1.8(0.3)	3.0(0.3)	5.0(0.4)	9.2(0.6)	20.6(0.7)
16	1.8(0.3)	3.0(0.3)	5.0(0.4)	9.3(0.6)	21.3(0.7)
∞	1.8(0.3)	3.0(0.3)	5.0(0.4)	9.3(0.6)	21.3(0.7)

Model 2. From (21), (18) and (19) we computed the optimum Q (using the same procedure as in Model 1) for $d = 0.05(0.05)0.20$, $\rho = 0.6(0.1)0.9$ and $r = 3$ and 4. With $r = 2$, the per cent gain in efficiency and the optimum Q are the same for both models. Table 2 gives the per cent gain in efficiency and the optimum Q for the selected values of d, ρ and r. Also, the per cent gain in efficiency and the optimum Q for Model 1 with $m = \infty$ are given for comparison.

Table 2. Per cent gain in efficiency of $\bar{x}'_0$ over $\bar{x}_0$, and in parantheses optimum Q

d \ ρ	0.6	0.7	0.8	0.9
		r = 3		
0.05	8.7(0.4)	15.4(0.5)	28.7(0.6)	56.1(0.7)
0.10	8.3(0.4)	14.4(0.4)	25.0(0.5)	46.8(0.6)
0.15	8.0(0.3)	13.5(0.4)	22.9(0.5)	41.2(0.6)
0.20	7.6(0.3)	12.6(0.4)	20.8(0.5)	36.1(0.6)
Model 1	7.3(0.3)	12.4(0.4)	22.4(0.5)	47.0(0.6)
		r = 4		
0.05	8.0(0.4)	14.3(0.5)	25.9(0.6)	49.3(0.7)
0.10	7.1(0.4)	12.3(0.5)	21.2(0.6)	37.0(0.7)
0.15	6.3(0.4)	10.7(0.4)	17.9(0.5)	29.9(0.6)
0.20	5.5(0.4)	9.8(0.4)	15.8(0.5)	25.0(0.6)
Model 1	5.8(0.3)	9.9(0.4)	17.8(0.5)	40.0(0.7)

Table 2 shows that model 1 is fairly robust to moderate deviations as shown by the alternate arithmetic model calculations, and the optimum Q for models 1 and 2 are either the same or deviate only by 0.1.

4.3 Variance of $\bar{x}_0'$ for the Current Population Survey rotation pattern

As described in Section 1, the rotation pattern for the Current Population Survey is as follows: eight rotation groups each of n_2 units are identified for each sample and a rotation group stays in the sample for four consecutive months, leaves the sample during the eight succeeding months, returns for another four consecutive months and then drops out. Therefore, the sample size is $n' = 8n_2 = 2n$, $r = 4$ and $m = 8$. On any occasion, $4n_2$ of the units in the sample are in the first cycle and the remaining $4n_2$ units are in the second cycle (since there are only two cycles for each rotation group). The composite estimator $\bar{x}_0'$ can be written as

$$\bar{x}_0' = \sum_{\alpha=0}^{-\infty} \sum_{k=1}^{N} v_{\alpha,k}\, x_{\alpha,k}$$

with variance

$$V(\bar{x}_0') = \left[N\, E(v_{0,k}^2) - \frac{1}{N}\right] S_0^2 + N \sum_{\alpha=-1}^{-\infty} E(v_{\alpha,k}^2)\, S_\alpha^2$$

$$+ N \sum_{\substack{\alpha\neq\alpha' \\ =0}}^{-\infty} E(v_{\alpha,k}\, v_{\alpha' k})\, S_{\alpha,\alpha'} \qquad (22)$$

where $v_{\alpha,k} = w_{\alpha,k}/2$ and $w_{\alpha,k}$ are the weights defined in (7) and (8). Also, $E(v_{\alpha,k}^2) = \frac{1}{2} E(w_{\alpha,k}^2)$, $E(v_{\alpha,k}\, v_{\alpha' k}) = \frac{1}{2} E(w_{\alpha,k}\, w_{\alpha' k})$ for $\alpha' = \alpha-1, \alpha-2, \alpha-3$; $E(v_{\alpha,k}\, v_{\alpha' k}) = \frac{1}{4} E(w_{\alpha,k}\, w_{\alpha' k})$ for $\alpha' = \alpha-9, \ldots, \alpha-15$; $\alpha\neq\alpha' = 0, -1, \ldots, -\infty$, and all other expectations are zero. Therefore, using the expressions for $E(w_{\alpha,k}\, w_{\alpha' k})$ and $E(v_{\alpha,k}\, v_{\alpha' k})$ the variance of $\bar{x}_0'$ is easily obtained from (22).

5. VARIANCE OF d_0' AND OPTIMUM Q

5.1 Variance of d_0'

The composite estimator of the change, $\bar{X}_0 - \bar{X}_{-1}$, is $d_0' = \bar{x}_0' - \bar{x}_{-1}'$

with variance

$$V(d_0') = V(\bar{x}_0') + V(\bar{x}_{-1}') - 2\,\mathrm{Cov}(\bar{x}_0', \bar{x}_{-1}').$$

The derivation of $V(\bar{x}_{-1}')$ and $\mathrm{Cov}(\bar{x}_0', \bar{x}_{-1}')$ is similar to that of $V(\bar{x}_0')$. Hence, without derivation, we have given the general formula for $V(d_0')$ in Appendix C. For Model 1 the general formula reduces to

$$V(d_0') = 2\left(\frac{1}{n} - \frac{1}{N}\right)(1 - \rho)\, S_0^2 - \frac{2n_2^3\, S_0^2\,(1 - Q)}{n^2 n_1^2 (1 - Q\rho)^2\,(1 - Q^2)\;\; 1 - (Q\rho)^{r+m}} \cdot$$

$$\Big\{ - (r-1)^2\rho + (2r^2-5r+3)\, Q\rho + r(r-1)\, Q\rho^2 + (2r^2-r-3)\, Q^2\rho$$

$$- r(3r-4)\, Q^2\rho^2 - r(2r-3)\, Q^2 - rQ^3 + (3r^2-2r+1)\, Q^3\rho$$

$$- r(3r-2)\, Q^3\rho^2 + r(r-1)\, Q^3\rho^3 - r(r-1)\, Q^4\rho^2 + r(r-1)\, Q^4\rho^3$$

$$+ Q^r\rho^{r-1}\Big[r(r-1)\,\rho - (r-1)\,\rho^2 - r^2 Q - r(r-3)\, Q\rho - (r^2-3r+3)\, Q\rho^2$$

$$+ r^2Q^2 + r(r-3)\, Q^2\rho + (r^2-3r+3)\, Q^2\rho^2 - r(r-1)\, Q^3\rho + (r-1)\, Q^3\rho^2 \Big]$$

$$+ Q^m\rho^{m+1} \Big[(r-1)^2 - (2r^2-5r+3)\, Q - r(r-1)\, Q\rho - 3(r-1)\, Q^2$$

$$+ 3r(r-1)\, Q^2\rho + (r^2-r-1)\, Q^3 - r(r-3)\, Q^3\rho - r^2Q^3\rho^2 + rQ^4$$

$$- r(r+1)\, Q^4\rho + r^2Q^4\rho^2 \Big] + Q^{m+r}\, \rho^{m+r} \Big[-r(r-1) + (r-1)\,\rho$$

$$+ 3r(r-1)\, Q - 3(r-1)\, Q\rho + r^2Q^2 - (6r^2 - 7r + 3)\, Q^2\rho + r(2r-1)\, Q^2\rho^2$$

$$- r^2Q^3 - (r^2-2r-1)\, Q^3\rho + r(4r-5)\, Q^3\rho^2 - r(r-1)\, Q^3\rho^3 + r(r-1)\, Q^4\rho$$

$$- r(r-1)\, Q^4\rho^3 \Big]\Big\} . \qquad (23)$$

The special case $x_{\alpha,k} = x_{\alpha'k}$ provides a check on (23). Then

$$d_0' = (1 - Q) \sum_{\alpha=0}^{-\infty} Q^{-\alpha}\,(\bar{x}_\alpha - \bar{x}_{\alpha-1})$$

and, using the formulae for $\mathrm{Cov}(\bar{x}_\alpha, \bar{x}_{\alpha'})$ given in sub-section 4.1, we obtain after simplification

$$V(d_0') = \frac{2n_2 S_0^2\,(1-Q)^2}{n^2(1-Q^2)(1-Q^{r+m})}\,(1 - Q^r - Q^m + Q^{r+m}) . \qquad (24)$$

With $\rho = 1$, (23) reduces to (24), thus providing a check.

The per cent gain in efficiency of d_0' over the estimator d_0 based on the difference between the sample means of two independent samples on occasions 0 and -1 is

$$\frac{V(d_0) - V(d_0')}{V(d_0')} \times 100 \tag{25}$$

where

$$V(d_0) = 2(\frac{1}{n} - \frac{1}{N})\, S_0^2 \;. \tag{26}$$

Since the results in sub-section 4.2 indicate that model 1 is fairly robust to moderate deviations, we did not investigate model 2 for change estimation.

5.2 Optimum Q and per cent gain in efficiency

As in the case of the current estimate, we ignored the term involving $1/N$ in (23) and (26) and computed the per cent gain in efficiency from (25) for $Q = 0.1(0.1)0.9$, $\rho = 0.5(0.1)0.9$, $r = 2(1)4(2)8$ and $m = sr$ with $s = 1(1)4, \infty$. The optimum Q for each combination (r, ρ, m) is that value of Q which maximizes the per cent gain in efficiency. Table 3 gives the per cent gain in efficiency of d_0' over d_0 and the optimum Q for d_0'. The per cent gain in efficiency of d_0' over d_0 and the optimum Q for $\bar{x}_0'$ from Table 1 is also included in Table 3. It may be noted that in most situations we are forced to use the same value of Q for both $\bar{x}$ and d_0'. If the current occasion estimate is more important, we may prefer to use the current occasion optimum Q for both d_0' and $\bar{x}_0'$.

Table 3 shows that considerable gains in efficiency of d_0' over d_0 can be obtained using the optimum Q for d_0', even with moderately large ρ. Also the optimum Q for $\bar{x}_0'$ leads to substantial gains in efficiency of d_0' over d_0, though the gains with the optimum Q for d_0' are larger, particularly for

Table 3. Per cent gain in efficiency of d_0' over d_0 using optimum Q for d_0' and $\bar{x}_0'$, and in parantheses optimum Q

m \ ρ	0.5		0.6		0.7		0.8		0.9	
					r = 2					
2	50.3(0.3)	47.4(0.2)	76.3(0.4)	65.2(0.2)	119.7(0.5)	103.4(0.3)	205.9(0.6)	174.9(0.4)	461.3(0.8)	354.8(0.5)
4	49.8(0.3)	47.1(0.2)	74.9(0.4)	64.5(0.2)	116.6(0.5)	100.7(0.3)	200.7(0.7)	166.8(0.4)	454.6(0.8)	327.3(0.5)
6	49.8(0.3)	47.1(0.2)	74.8(0.4)	64.5(0.2)	116.3(0.5)	100.6(0.3)	199.8(0.7)	166.0(0.4)	451.9(0.8)	383.4(0.6)
8	49.8(0.3)	47.1(0.2)	74.8(0.4)	64.5(0.2)	116.2(0.5)	100.6(0.3)	199.5(0.7)	166.0(0.4)	450.7(0.8)	381.5(0.6)
∞	49.8(0.3)	47.1(0.2)	74.8(0.4)	64.5(0.2)	116.2(0.5)	100.6(0.3)	199.4(0.7)	166.0(0.4)	449.5(0.8)	380.6(0.6)
					r = 3					
3	66.8(0.4)	62.6(0.2)	100.4(0.5)	94.1(0.3)	156.6(0.6)	145.0(0.4)	268.9(0.7)	242.4(0.5)	601.7(0.8)	503.3(0.6)
6	66.7(0.4)	62.6(0.2)	100.0(0.5)	93.9(0.3)	155.5(0.6)	144.1(0.4)	266.1(0.7)	238.3(0.5)	599.8(0.9)	482.9(0.6)
9	66.7(0.4)	62.6(0.2)	100.0(0.5)	93.9(0.3)	155.5(0.6)	144.0(0.4)	265.8(0.8)	238.0(0.5)	599.6(0.9)	479.9(0.6)
∞	66.7(0.4)	62.6(0.2)	100.0(0.5)	93.9(0.3)	155.5(0.6)	144.0(0.4)	265.8(0.8)	238.0(0.5)	595.9(0.9)	479.3)0.6)
					r = 4					
4	75.0(0.4)	71.1(0.2)	112.4(0.5)	106.4(0.3)	175.2(0.7)	163.1(0.4)	301.2(0.8)	270.5(0.5)	677.4(0.9)	555.2(0.6)
8	75.0(0.4)	71.1(0.2)	112.3(0.5)	106.3(0.3)	175.0(0.7)	162.8(0.4)	300.7(0.8)	268.9(0.5)	676.9(0.9)	608.1(0.7)
12	75.0(0.4)	71.1(0.2)	112.3(0.5)	106.3(0.3)	175.0(0.7)	162.8(0.4)	300.6(0.8)	268.9(0.5)	676.7(0.9)	606.0(0.7)
∞	75.0(0.4)	71.1(0.2)	112.3(0.5)	106.3(0.3)	175.0(0.7)	162.8(0.4)	300.6(0.8)	268.9(0.5)	676.7(0.9)	605.0(0.7)
					r = 6					
6	83.3(0.4)	82.3(0.3)	124.9(0.6)	119.7(0.3)	194.7(0.7)	183.9(0.4)	334.4(0.8)	319.7(0.6)	752.5(0.9)	685.7(0.7)
12	83.3(0.4)	82.3(0.3)	124.9(0.6)	119.7(0.3)	194.6(0.7)	183.9(0.4)	334.1(0.8)	319.3(0.6)	752.1(0.9)	680.0(0.7)
∞	83.3(0.4)	82.3(0.3)	124.9(0.6)	119.7(0.3)	194.6(0.7)	183.9(0.4)	334.1(0.8)	319.3(0.6)	752.0(0.9)	679.6(0.7)
					r = 8					
8	87.4(0.4)	86.6(0.3)	131.2(0.6)	126.8(0.3)	204.3(0.7)	195.3(0.4)	350.6(0.8)	337.8(0.6)	789.5(0.9)	726.5(0.7)
16	87.4(0.4)	86.6(0.3)	131.2(0.6)	126.8(0.3)	204.3(0.7)	195.3(0.4)	350.6(0.8)	337.7(0.6)	789.1(0.9)	724.4(0.7)
∞	87.4(0.4)	86.6(0.3)	131.2(0.6)	126.8(0.3)	204.3(0.7)	195.3(0.4)	350.6(0.8)	337.7(0.6)	789.1(0.9)	724.3(0.7)

high ρ. Unlike the current occasion case where the optimum r is 2, the gain in efficiency of d_0' over d_0 increases with r. Therefore, if both current occasion and change are of interest, we may prefer to use an r other than 2. The variance of d_0' for moderate and large values of m differs very little from the variance of d_0' with $m = \infty$. The optimum Q for d_0' is unaffected by the value of m in most of the cases.

6. CONCLUDING REMARKS

In many practical situations, the exponential correlation pattern (model 1) may be quite reasonable. Also, a comparison with the arithmetic correlation pattern (model 2) showed that the exponential correlation pattern was fairly robust to moderate deviations. However, in a monthly survey with characteristics strongly influenced by seasonal variations, the correlation between occasions 12 months apart may be about the same magnitude (or even larger) as compared to the correlation between consecutive occasions. In such situations, model 3,

$$S_{\alpha,\alpha-12j-i} = S_{0,-12j-i} = \rho_1^i \rho_2^j S_0^2, \qquad i = 0, 1, \ldots, 11; \; j = 0, 1, 2, \ldots$$

where ρ_1 is the correlation between consecutive occasions and ρ_2 is the correlation between occasions 12 months apart, may be more appropriate. Then, it is necessary to generalize the composite estimators $\bar{x}_0'$ and d_0' and construct composite estimators which take advantage of both ρ_1 and ρ_2 explicitly. This problem has been investigated and the results will be published in a later paper. The generalized composite estimators lead to considerable gains in efficiency over $\bar{x}_0$ and d_0', when ρ_1 and ρ_2 are of approximately the same magnitude.

In large scale surveys multistage sampling is often employed. In a two-stage design, rotation sampling is commonly used within each primary, and, therefore, the composite estimators reduce the within primary variation only. It may be reasonable to assume that the correlation of secondaries between occasions is approximately the same in each primary. Then we could use the tables for the optimum Q given here and construct composite estimators, with the same Q, within each selected primary, and then determine the composite estimator of the over-all population mean. For the estimator of variance, we could use the estimator based on the mean square of the within primary composite estimators (this estimator of variance is unbiased if the primaries are selected with replacement; if the primaries are selected without replacement it over-estimates the variance).

We wish to thank Professor H. O. Hartley for helpful suggestions.

APPENDIX A

The expectations $E(w_{\alpha,k}\, w_{\alpha' k})$

The expectations $NE(w_{0,k}\, w_{\alpha,k})$, $\alpha \leq -1$, are as follows:

$$NE(w_{0,k}\, w_{-(s+1)(r+m),k}) = \frac{n_2^2}{nn_1^2}(1 + \frac{n_1}{n_2} Q)Q\, Q^{s(r+m)}, \quad s = 0, 1, \ldots \infty$$

$$NE(w_{0,k}\, w_{-s(r+m)-1,k}) = -\frac{n_2}{n^2}(1 + \frac{n_2}{n_1} Q)^2 \quad Q^{s(r+m)+1}$$

$$NE(w_{0,k}\, w_{-s(r+m)-t,k}) = \frac{n_2^2}{n^2 n_1}(1 + \frac{n_2}{n_1} Q)\left[t(1 - Q) - r \right] Q^{s(r+m)+t}$$
$$t = 2, \ldots, r-2.$$

$$NE(w_{0,k}\, w_{-s(r+m)-r+1,k}) = -\frac{n_2}{n^2}(1 + \frac{n_2}{n_1} Q)(Q + \frac{n_2}{n_1})\, Q^{s(r+m)+r-1}$$

$$NE(w_{0,k}\, w_{-s(r+m)-m-1,k}) = \frac{n_2}{n^2}(1 + \frac{n_2}{n_1} Q)(1 - Q)\, Q^{s(r+m)+m+1}$$

$$NE(w_{0,k}\, w_{-s(r+m)-m-t,k}) = \frac{n_2^2}{n^2 n_1} \left[(1 + 2\frac{n_2}{n_1} Q - \frac{n_2}{n_1} Q^2)\frac{n}{n_2} + (Q - 1)(1 + \frac{n_2}{n_1} Q)\, t \right] Q^{s(r+m)+m+t}$$

$$t = 2, \ldots, r-1.$$

$$NE(w_{0,k}\, w_{-s(r+m)-r-t,k}) = 0, \qquad t = 0, \ldots, m-r.$$

The expectations $NE(w_{\alpha,k}\, w_{\alpha' k})$, α, $\alpha' \leq 1$, $(\alpha \neq \alpha')$, are as follows:

$$NE(w_{\alpha,k}\, w_{\alpha-s(r+m),k}) = \frac{n_2}{nn_1} (Q^2 + 2\frac{n_2}{n_1} Q + 1)\, Q^{s(r+m)-2\alpha}$$

$$NE(w_{\alpha,k}\, w_{\alpha-s(r+m)-t,k}) = -\frac{n_2^3}{n^2 n_1^2} (1 - Q)^2\, t\, Q^{s(r+m)-2\alpha+t}$$

$$t = 1, \ldots, r-2$$

$$NE(w_{\alpha,k}\, w_{\alpha-s(r+m)-r+1,k}) = -\frac{n_2}{n^2} (1 + \frac{n_2}{n_1} Q)(Q + \frac{n_2}{n_1})\, Q^{s(r+m)-2\alpha+r-1}$$

$$NE(w_{\alpha,k}\, w_{\alpha-s(r+m)-m-1,k}) = -\frac{n_2}{n^2} (1 + \frac{n_2}{n_1} Q)(Q + \frac{n_2}{n_1})\, Q^{s(r+m)-2\alpha+m+1}$$

$$NE(w_{\alpha,k}\, w_{\alpha-s(r+m)-m-t,k}) = -\frac{n_2^3}{n^2 n_1^2} (r - t)(1 - Q)^2\, Q^{s(r+m)-2\alpha+m+t}$$

$$t = 2, \ldots, r-1$$

$$NE(w_{\alpha,k}\, w_{\alpha-s(r+m)-r-t,k}) = 0, \qquad t = 0, \ldots, m-r.$$

APPENDIX B

Variance of $\bar{x}_0'$

The variance of $\bar{x}_0'$ for the general rotation pattern is

$$V(\bar{x}_0') = (\frac{1}{n} - \frac{1}{N})\, S_0^2 + \frac{n_2}{nn_1} Q^2 S_0^2 + 2\frac{n_2}{n^2} (1 + \frac{n_2}{n_1} Q)(1 - Q)\, Q^{m+1} \cdot$$

$$\sum_{s=0}^{\infty} Q^{s(r+m)} S_{0,-s(r+m)-m-1} + 2\frac{n_2^2}{nn_1^2} (1 + \frac{n_1}{n_2} Q) \sum_{s=1}^{\infty} Q^{s(r+m)+1} S_{0,-s(r+m)}$$

$$+ 2 \frac{n_2^2}{n^2 n_1} \sum_{t=2}^{r-1} Q^{m+t} \left[(1 + 2 \frac{n_2}{n_1} Q - \frac{n_2}{n_1} Q^2) \frac{n}{n_2} - (1 - Q)(1 + \frac{n_2}{n_1} Q) t \right] .$$

$$\sum_{s=0}^{\infty} Q^{s(r+m)} S_{0,-s(r+m)-m-t} + 2 \frac{n_2^2}{n^2 n_1} (1 + \frac{n_2}{n_1} Q) \sum_{t=1}^{r-1} Q^t \left[t(1 - Q) - r \right] .$$

$$\sum_{s=0}^{\infty} Q^{s(r+m)} S_{0,-s(r+m)-t} + \frac{n_2}{nn_1} (Q^2 + 2 \frac{n_2}{n_1} Q + 1) \sum_{\alpha=-1}^{-\infty} Q^{-2\alpha} S_{\alpha}^2$$

$$+ 2 \frac{n_2}{nn_1} (Q^2 + 2 \frac{n_2}{n_1} Q + 1) \sum_{\alpha=-1}^{-\infty} Q^{-2\alpha} \sum_{s=1}^{\infty} Q^{s(r+m)} S_{\alpha,\alpha-s(r+m)} - 2 \frac{n_2^3}{n^2 n_1^2} (1 - Q)^2 .$$

$$\sum_{t=1}^{r-2} t \sum_{\alpha=-1}^{-\infty} Q^{-2\alpha+t} \sum_{s=0}^{\infty} Q^{s(r+m)} S_{\alpha,\alpha-s(r+m)-t} - 2 (\frac{n_2}{n^2} Q)(Q + \frac{n_2}{n_1})(1 + \frac{n_2}{n_1} Q) .$$

$$\sum_{\alpha=-1}^{-\infty} Q^{-2\alpha+r-1} \sum_{s=0}^{\infty} Q^{s(r+m)} S_{\alpha,\alpha-s(r+m)-r+1} - 2 \frac{n_2}{n^2} (1 + \frac{n_2 Q}{n_1})(Q + \frac{n_2}{n_1}) .$$

$$\sum_{\alpha=-1}^{-\infty} Q^{-2\alpha+m+1} \sum_{s=0}^{\infty} Q^{s(r+m)} S_{\alpha,\alpha-s(r+m)-m-1} - 2 \frac{n_2^3}{n^2 n_1^2} (1 - Q)^2 \sum_{t=2}^{r-1} (r - t) .$$

$$\sum_{\alpha=-1}^{-\infty} Q^{-2\alpha+m+t} \sum_{s=0}^{\infty} Q^{s(r+m)} S_{\alpha,\alpha-s(r+m)-m-t}$$

This formula is valid for $r > 2$. For $r = 2$, replace the terms involving $\sum_{1}^{r-1}$ and $\sum_{2}^{r-2}$ by zero.

APPENDIX C

Variance of d'_0

The variance of d'_0 for the general rotation pattern is

$$V(d'_0) = \frac{(1 - Q)^2}{Q^2} V(\bar{x}'_0) + \frac{(2Q - 1)}{Q^2}\Big[(\frac{1}{n} - \frac{1}{N}) S_0^2 + Q^2(\frac{1}{n_1} - \frac{1}{n}) S_0^2$$

$$+ Q^2(\frac{1}{n_1} - \frac{1}{N}) S_{-1}^2 - 2Q(\frac{1}{n} - \frac{1}{N}) S_{0,-1} - 2Q^2(\frac{1}{n_1} - \frac{1}{n}) S_{0,-1} \Big]$$

$$- \frac{2(1 - Q)}{Q^2} \sum_{s=0}^{\infty} Q^{s(r+m)} \sum_{t=1}^{r-1} Q^t .$$

$$\Big\{ \frac{n_2}{n^2 n_1^2} (n_1 + n_2 Q) \big[(r - t) n_1 - tn_2 Q\big] S_{0,-s(r+m)-t}$$

$$- \frac{1}{nn_1^2} Q(n_1 + n_2 Q)(n_1 - tn_2) .$$

$$S_{0,-s(r+m)-t-1} + \frac{1}{n_1^2} Q^2(n_1 - tn_2) S_{-1,-s(r+m)-t-1}$$

$$- \frac{1}{nn_1^2} Q \big[nn_1 - tn_2(n_1 + n_2 Q)\big] .$$

$$S_{-1,-s(r+m)-t} \Big\} + \frac{2(1 - Q)}{NQ^2} \sum_{s=1}^{\infty} Q^2(S_{0,-s} - QS_{0,-s-1} - QS_{-1,-s}$$

$$+ Q^2 S_{-1,-s-1})$$

$$- \frac{2(1 - Q)}{Q^2} \sum_{s=0}^{\infty} Q^{s(r+m)} \sum_{t=1}^{r-1} Q^{m+t} \Big\{ \frac{n_2}{n^2 n_1^2} (n_1 + n_2 Q)\big[t(n_1 + n_2 Q) - nQ\big] .$$

$$S_{0,-s(r+m)-m-t}$$

$$- \frac{n_2}{nn_1^2} Q \big[t(n_1 + n_2 Q) - nQ \big] S_{0,-s(r+m)-m-t-1} + \frac{n_2}{n_1^2} (t - 1) Q^2 .$$

$$S_{-1,-s(r+m)-m-t-1}$$

$$- \frac{n_2}{nn_1^2} (t - 1) Q(n_1 + n_2Q) S_{-1,-s(r+m)-m-t} \Big\} - \frac{2(1 - Q)}{Q^2} \sum_{s=1}^{\infty} Q^{s(r+m)} \cdot$$

$$\Big\{ \frac{1}{nn_1} (n_1 + n_2Q^2) S_{0,-s(r+m)} - \frac{1}{nn_1} Q(n_1 + n_2Q) S_{0,-s(r+m)-1}$$

$$+ \frac{1}{n_1} Q^2 S_{-1,-s(r+m)-1} - \frac{1}{nn_1} Q(n_1 + n_2Q) S_{-1,-s(r+m)} \Big\} .$$

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